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Intersection of finitely generated congruences over term algebra

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Abstract

We show that it is decidable for any given ground term rewrite systems R and S if there is a ground term rewrite system U such that $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$. If the answer is yes, then we can effectively construct such a ground term rewrite system U. In other words, for any given finitely generated congruences ρ and τ over the term algebra, it is decidable if $\rho \cap \tau$ is a finitely generated congruence. If the answer is yes, then we can effectively construct a ground term rewrite system U such that $\leftrightarrow_U^* = \rho \cap \tau$.

Keywords: Finitely generated congruence; Term algebra; Ground term rewrite system; Tree automaton

1. Introduction

Vágvölgyi [19] studied the intersection of two finitely generated congruences over the term algebra, and showed the following.

Proposition 1.1 (Vágvölgyi [19]). *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ such that $trunk(\leftrightarrow_R^*) = trunk(\leftrightarrow_S^*)$. Then one can effectively construct a gtrs U over Σ such that $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$.*

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Küsters and Borgida [13] studied the intersection of two finitely generated right congruences over strings. They showed that given two finite generating systems, it is decidable if the intersection of the generated right congruences is finitely generated.

We generalize the above results. We show that it is decidable for any given ground term rewrite systems R and S if there is a ground term rewrite system U such that $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$. If the answer is yes, then we can effectively construct such a ground term rewrite system U. In other words, for any given finitely generated congruences ρ and τ over the term algebra, it is decidable if $\rho \cap \tau$ is a finitely generated congruence.

We now present the main concepts and a sketch of the paper. We adopt the trunk of a congruence ρ on the term algebra **TA** from [6]. For a reduced ground term rewrite system R, $trunk(\leftrightarrow_R^*)$ consists of all trees which can be reduced by R to some subtree of the left-hand side or the right-hand side of some rule. Let ρ and τ be arbitrary congruences on **TA**. We extend ranked alphabet Σ into a ranked alphabet Δ by adding new nullary symbols. Intuitively, these nullary symbols stand for binary relations over T_Σ, and a tree in T_Δ can be interpreted as a binary relation over T_Σ. Then we introduce the tree language $\rho \diamond \tau$ over Δ, which we call the *cooperation between* ρ and τ *in inducing* $\rho \cap \tau$ *in* $trunk(\rho)$ *beyond* $trunk(\tau)$. A tree in $\rho \diamond \tau$ stands for a subset of $\rho \cap \tau$. For reduced ground term rewrite systems R and S, $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ stands for all pairs $(p,q) \in \leftrightarrow_R^* \cap \leftrightarrow_S^*$ such that $p, q \in trunk(\leftrightarrow_R^*)$ and $p, q \notin trunk(\leftrightarrow_S^*)$.

We show the following five statements.

Statement 1.2. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. Then tree language $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is recognizable.*

Statement 1.3. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. Then it is decidable if $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is finite.*

Statement 1.4. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. If $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is infinite or $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ is infinite, then there is no gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$.*

Statement 1.5. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. If both $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ and $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ are finite, then we can effectively construct a gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$.*

Assume that $trunk(\rho) = trunk(\tau)$. Then $\rho \diamond \tau = \emptyset$ and $\tau \diamond \rho = \emptyset$. Hence by Statement 1.5 $\rho \cap \tau$ is finitely generated. Thus Statement 1.5 is a generalization of Proposition 1.1. Statements 1.3, 1.4, and 1.5 imply our main decidability result.

Statement 1.6. *It is decidable for any given ground term rewrite systems R and S if there is a ground term rewrite system U such that $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$. If the answer is yes, then we can effectively construct such a ground term rewrite system U.*

In Section 2 we carefully review all notions, notations and preliminary results used in the paper. In Section 2.1 we carefully recall and discuss the results on ground term rewriting and tree automata of Fülöp and Vágvölgyi, see [5,7]. In Section 2.2 we introduce tree language $\rho\diamond\tau$. In Section 2.3 we illustrate the above concepts by three examples. In Section 2.4 we fix some notations used throughout the paper, and introduce five types of mappings evaluating trees in T_Δ.

In Section 3 we show Statement 1.2. This implies Statement 1.3. In Section 4 we show Statement 1.4. In Section 5 we show Statement 1.5. In Section 6 we show our main decidability result, Statement 1.6.

2. Preliminaries

In this section we present a brief review of the notions, notations and preliminary results used in the paper.

Relations. A relation over a set A is a subset $\rightarrow$ of $A\times A$. We write $a\rightarrow b$ for $(a,b)\in\rightarrow$. We denote by $\rightarrow^*$ the reflexive, transitive closure and by $\leftrightarrow^*$, the reflexive, symmetric, and transitive closure of $\rightarrow$. Note that $\leftrightarrow^*$ is an equivalence relation.

A relation $\rightarrow$ is called

- Noetherian if there exists no infinite sequence of elements $a_1,a_2,a_3,\dots$ in A such that $a_1\rightarrow a_2\rightarrow a_3\rightarrow\cdots$,
- confluent if for any elements a_1,a_2,a_3 in A, whenever $a_1\rightarrow^* a_2$ and $a_1\rightarrow^* a_3$, there exists an element a_4 in A such that $a_2\rightarrow^* a_4$ and $a_3\rightarrow^* a_4$,
- convergent if it is Noetherian and confluent.

Let $\rightarrow$ be a relation over a set A. An element $a\in A$ is irreducible with respect to $\rightarrow$ if there exists no $b\in A$ such that $a\rightarrow b$. It is well-known that for any convergent relation $\rightarrow$ and any class Z of $\leftrightarrow^*$, Z contains exactly one irreducible element a, and that for any element b in the class Z, $b\rightarrow^* a$. We call a the $\rightarrow$-normal form of b.

Let ρ be an equivalence relation on A. Then for every $a\in A$, we denote by $[a]_\rho$ the ρ-class containing a, i.e. $[a]_\rho=\{b\mid a\rho b\}$. We say that ρ is *of finite index* if the set $\{[a]_\rho\mid a\in A\}$ is finite. Let H be a set of ρ-classes, then by $\bigcup H$ we mean $\bigcup(C\mid C\in H)$.

Terms. A ranked alphabet Σ is a finite set of symbols in which every element has a unique rank in the set of nonnegative integers. For each integer $m\geqslant 0$, Σ_m denotes the elements of Σ which have rank m.

Let Y be a set. The set of terms over Σ with Y is the smallest set U for which

(i) $\Sigma_0\cup Y\subseteq U$ and

(ii) $f(t_1,\dots,t_m)\in U$ whenever $f\in\Sigma_m$ with $m\geqslant 1$ and $t_1,\dots,t_m\in U$.

Terms are also called trees. We need a countably infinite set $X=\{x_1,x_2,\dots\}$ of variable symbols kept fixed throughout the paper. The set of the first n elements $x_1,\dots,x_n$ of X is denoted by X_n. The set $T_\Sigma(\emptyset)$ is written simply as T_Σ and called the set of ground trees over Σ. We distinguish a subset $\bar{T}_\Sigma(X_n)$, $n\geqslant 0$, of $T_\Sigma(X_n)$ as follows: a tree $t\in T_\Sigma(X_n)$ is in $\bar{T}_\Sigma(X_n)$ if and only if each variable symbol of X_n appears exactly once in t. For example, if $\Sigma=\Sigma_0\cup\Sigma_2$ with $\Sigma_0=\{\#\}$ and $\Sigma_2=\{f\}$, then $f(x_1,f(\#,x_1))\in T_\Sigma(X_1)$ but $f(x_1,f(\#,x_1))\notin\bar{T}_\Sigma(X_1)$. Furthermore, $f(x_2,f(\#,x_1))\in\bar{T}_\Sigma(X_2)$.

For a ground term $t \in T_\Sigma$, the set $sub(t)$ of subtrees of t is defined by recursion as follows:

(i) if $t \in \Sigma_0$, then $sub(t) = \{t\}$,

(ii) if $t = f(t_1, \ldots, t_m)$ for some $m \geqslant 1$, $f \in \Sigma_m$, and $t_1, \ldots, t_m \in T_\Sigma$, then we have $sub(t) = \bigcup(sub(t_i) \mid 1 \leqslant i \leqslant m) \bigcup \{t\}$.

For a tree language $L \subseteq T_\Sigma$, the set $sub(L)$ of subtrees of elements of L is defined by the equation $sub(L) = \bigcup(sub(t) \mid t \in L)$.

For a term $t \in T_\Sigma$, *the height of t* is denoted by $height(t)$ and is defined by recursion:

(i) if $t \in \Sigma_0$, then $height(t) = 0$, and

(ii) if $t = f(t_1, \ldots, t_m)$ with $m \geqslant 1$ and $f \in \Sigma_m$, then $height(t) = max\{height(t_i) \mid 1 \leqslant i \leqslant m\} + 1$.

Let N be the set of all positive integers. For a term $t \in T_\Sigma(X)$, *the set of occurrences* $\mathrm{O}(t) \subseteq N^*$ is defined by recursion:

(i) if $t \in \Sigma_0 \cup X$, then $\mathrm{O}(t) = \{\lambda\}$, and

(ii) if $t = f(t_1, \ldots, t_m)$ with $m \geqslant 1$ and $f \in \Sigma_m$, then $\mathrm{O}(t) = \{\lambda\} \cup \{i\alpha \mid 1 \leqslant i \leqslant m$ and $\alpha \in \mathrm{O}(t_i)\}$.

For any $t \in T_\Sigma$, let $size(t)$ be the cardinality of $\mathrm{O}(t)$.

For any $t \in T_\Sigma$ and $\alpha \in \mathrm{O}(t)$, we introduce the subterm $t/\alpha \in T_\Sigma$ of t at α as follows:

(i) for $t \in \Sigma_0$, we define $t/\lambda = t$;

(ii) for $t = f(t_1, \ldots, t_m)$ with $m \geqslant 1$ and $f \in \Sigma_m$, if $\alpha = \lambda$ then $t/\alpha = t$, otherwise, if $\alpha = i\beta$ with $1 \leqslant i \leqslant m$, then $t/\alpha = t_i/\beta$.

Finally, for any $t \in T_\Sigma$, $\alpha \in \mathrm{O}(t)$, and $r \in T_\Sigma$, we define $t[\alpha \leftarrow r] \in T_\Sigma$.

(i) If $\alpha = \lambda$, then $t[\alpha \leftarrow r] = r$.

(ii) If $\alpha = i\beta$, for some integer i, then $t = f(t_1, \ldots, t_m)$ with $f \in \Sigma_m$ and $1 \leqslant i \leqslant m$. Then $t[\alpha \leftarrow r] = f(t_1, \ldots, t_{i-1}, t_i[\beta \leftarrow r], t_{i+1}, \ldots, t_m)$.

A substitution is a mapping $\sigma : X \to T_\Sigma(X)$ which is different from the identity only for a finite subset $Dom(\sigma)$ of X. For any substitution σ, the term $\sigma(t)$ is produced from t by replacing each occurrence of x_i with $\sigma(x_i)$ for $i \geqslant 1$. For any trees $t \in \bar{T}(X_k)$, $t_1, \ldots, t_k \in T_\Sigma(X)$ and for the substitution σ with $Dom(\sigma) = X_k$, $k \geqslant 1$, and $\sigma(x_i) = t_i$ for $i = 1, \ldots, k$, we denote the term $\sigma(t)$ by $t[t_1, \ldots, t_k]$ as well. For any trees $t \in \bar{T}(X_k)$, $t_1, \ldots, t_n \in T_\Sigma(X)$ and for the substitution σ with $Dom(\sigma) = \{x_{i_1}, \ldots, x_{i_n}\} \subseteq X_k$, $n \geqslant 1$ and $\sigma(x_{i_j}) = t_j$ for $j = 1, \ldots, n$, we denote the term $\sigma(t)$ by $t[x_{i_1} \leftarrow t_1, \ldots, x_{i_k} \leftarrow t_k]$ as well. We say that a substitution $\sigma : X \to X$ is a variable renaming if σ is injective.

Algebras. Let Σ be a ranked alphabet. A Σ algebra is a system $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$, where B is a nonempty set, called the carrier set of $\mathbf{B}$, and $\Sigma^{\mathbf{B}} = \{f^{\mathbf{B}} \mid f \in \Sigma\}$ is a Σ-indexed set of operations over B such that for every $f \in \Sigma_m$ with $m \geqslant 0$, $f^{\mathbf{B}}$ is a mapping from B^m to B. We assign an element $t^{\mathbf{B}} \in B$ to every term $t \in T_\Sigma(B)$. Let $t = f(t_1, \ldots, t_m)$, $f \in \Sigma_m$, $m \geqslant 0$. Then $t^{\mathbf{B}} = f^{\mathbf{B}}(t_1^{\mathbf{B}}, \ldots, t_m^{\mathbf{B}})$. An equivalence relation $\rho \subseteq B \times B$ is a congruence on $\mathbf{B}$ if

$$f^{\mathbf{B}}(t_1, \ldots, t_m) \rho f^{\mathbf{B}}(p_1, \ldots, p_m)$$

whenever $f \in \Sigma_m$, $m \geqslant 0$, and $t_i \rho p_i$, for $1 \leqslant i \leqslant m$. For each $B' \subseteq B$, let $[B']_\rho = \{[b]_\rho \mid b \in B'\}$. The least congruence on $\mathbf{B}$ containing a given relation $\sigma \subseteq B \times B$ is called the *congruence generated by* σ. A congruence on $\mathbf{B}$ is finitely generated if it is generated by a finite relation $\sigma \subseteq B \times B$. We define the *quotient algebra* $\mathbf{B}/\rho = ([B]_\rho, \Sigma^{\mathbf{B}/\rho})$ of the algebra $\mathbf{B}$ modulo the congruence ρ as follows. For all $f \in \Sigma_m$, $m \geqslant 0$, and $b_1, \dots, b_m \in B$, we put $f^{\mathbf{B}/\rho}([b_1]_\rho, \dots, [b_m]_\rho) = [f^{\mathbf{B}}(b_1, \dots, b_m)]_\rho$.

In this paper we shall mainly deal with the algebra $\mathbf{TA} = (T_\Sigma, \Sigma)$ of terms over Σ, where for $f \in \Sigma_m$ with $m \geqslant 0$ and $t_1, \dots, t_m \in T_\Sigma$, we have

$$f^{\mathbf{TA}}(t_1, \dots, t_m) = f(t_1, \dots, t_m).$$

We adopt the concepts of a simple class and of a compound class of a congruence ρ on the term algebra $\mathbf{TA}$ from [6]. Informally, these concepts are defined as follows. Clearly, every ρ-class Z can be written as the union of sets of the form $\{f(z_1, \dots, z_m) \mid z_i \in Z_i, 1 \leqslant i \leqslant m\}$ for some suitable function symbols f and ρ-classes $Z_1, \dots, Z_m$. Especially, if the union has only one member, i.e., $Z = \{f(z_1, \dots, z_m) \mid z_i \in Z_i, 1 \leqslant i \leqslant m\}$, then Z is called a simple class. If a class is not simple, then it is compound.

Formally, given a congruence ρ on $\mathbf{TA}$, a ρ-class Z is called simple if for any function symbols $f \in \Sigma_m$, $g \in \Sigma_n$, with $m, n \geqslant 0$ and ρ-classes $Z_1, \dots, Z_m$, $Z'_1, \dots, Z'_n$, if $f^{\mathbf{TA}/\rho}(Z_1, \dots, Z_m) = Z$ and $g^{\mathbf{TA}/\rho}(Z_1, \dots, Z_n) = Z$, then $f = g$, $m = n$, $Z_1 = Z'_1, \dots, Z_m = Z'_m$. If a ρ-class Z is not simple then it is called a compound class. The set of all compound classes is denoted by $comp(\rho)$.

Next we adopt the trunk of a congruence ρ from [6]. Let ρ be a congruence on $\mathbf{TA}$, the trunk $trunk(\rho)$ of ρ is the set $sub(\bigcup comp(\rho))$. We write $stub(\rho)$ for $[trunk(\rho)]_\rho$.

Ground term rewrite systems. A ground term rewrite system (gtrs) over a ranked alphabet Σ is a finite subset R of $T_\Sigma \times T_\Sigma$. The elements of R, called rules, can be used to define a relation, called rewriting relation, $\to_R$ introduced as follows: for any $p, q \in T_\Sigma$, we have $p \to_R q$ if and only if there exists a rule, (u, v) in R and a context $c \in \bar{T}_\Sigma(X_1)$ such that $p = c[u]$ and $q = c[v]$. The rules in R will be written in the form $u \to v$ as well. Moreover, we say that u is the left-hand side and v is the right-hand side of the rule $u \to v$. Besides the "one-way" relations $\to_R$ and $\to_R^*$ we also consider the congruence relation on $\mathbf{TA}$ generated by R, which is $\leftrightarrow_R^*$.

For a gtrs R over a ranked alphabet Σ, by the set of subterms occurring in R we mean the set

$$sub(R) = \bigcup \{sub(u) \cup sub(v) \mid u \to v \text{ is in } R\}.$$

We say that R is Noetherian, (confluent, etc.) if $\to_R$ is Noetherian (confluent, etc.). A term $t \in T_\Sigma$ is irreducible with respect to R if it is irreducible with respect to $\to_R$. A gtrs R is reduced if for every rule $u \to v$ in R, u is irreducible with respect to $R - \{u \to v\}$ and v is irreducible with respect to R.

We recall the following two important results.

Proposition 2.1 (Snyder [16]). *Any reduced gtrs R is convergent.*

We now recall a result on the trunk of a congruence generated by a reduced gtrs.

Proposition 2.2 (Vágvölgyi [18]). *Let R be a reduced gtrs over a ranked alphabet Σ. Then*

$$trunk\left(\overset{*}{\underset{R}{\leftrightarrow}}\right) = \bigcup \{[t]_{\leftrightarrow_R^*} \mid t \in sub(R)\}.$$

The following result was shown by presenting fast algorithms, see [8,16].

Proposition 2.3. *Let Σ be a ranked alphabet. For every ground term rewrite system R over Σ, one can effectively construct a reduced ground term rewrite system S over Σ such that $\leftrightarrow_R^* = \leftrightarrow_S^*$.*

For further results on gtrs's see [1,2,3,10,11,12,14,15].

Tree automata. Let Σ be a ranked alphabet. A tree automaton A over Σ is a gtrs over the ranked alphabet $\Sigma \cup STATES$, where $STATES$, the state set of A, consists of nullary function symbols, and $STATES \cap \Sigma = \emptyset$. Each rule in A is of the form

$$f(a_1,\dots,a_n) \to a, \quad \text{where } f \in \Sigma_n,\ n \geqslant 0,\ a, a_1,\dots,a_n \in STATES.$$

In case $n=0$ we write f rather than $f(\ )$. We say that a tree automaton A over Σ is deterministic if for any $f \in \Sigma_m$, $m \geqslant 0$, $a_1,\dots,a_m \in STATES$, there is at most one rule with left-hand side $f(a_1,\dots,a_m)$ in A.

A state $a \in STATES_A$ is reachable if there is a tree $t \in T_\Sigma$ such that $t \to_A^* a$. The following can be shown by applying well-known techniques of tree automaton theory, see [9].

Proposition 2.4. *Let A be a tree automaton over Σ. Let $a \in STATES_A$. It is decidable if the state a is reachable. Moreover, if a is reachable, then one can effectively construct a tree $t \in T_\Sigma$ such that $t \to_A^* a$.*

Proposition 2.5. *Let Σ be a ranked alphabet, and let A be a deterministic tree automaton over Σ. Then A is a reduced gtrs over the ranked alphabet $\Sigma \cup STATES$.*

Proof. By direct inspection of the rules of A. $\square$

The tree language recognized by a state set $Q \subseteq STATES$ of A is

$$L(A,Q) = \left\{ t \in T_\Sigma \mid t \xrightarrow[A]{*} a \text{ for some } a \in Q \right\}.$$

When Q is a singleton, i.e., $Q=\{q\}$, we simply write $L(A,q)$ rather than $L(A,\{q\})$.

We say that a tree language L over Σ is *recognizable* if L is recognized by a state set $Q \subseteq STATES$ of some tree automaton A.

Proposition 2.6. *For any tree automaton A over Σ and any state set $Q \subseteq STATES_A$ it is decidable if $L(A,Q)$ is finite.*

By Proposition 2.3, Proposition 2.2, and by the results of Brainerd [1], Kozen [11], Fülöp and Vágvölgyi [4,17], we have the following result.

Proposition 2.7. *For any gtrs R over a ranked alphabet Σ, trunk$(\leftrightarrow_R^*)$ is recognizable. Moreover, one can effectively construct a tree automaton A over Σ and a state set $Q \subseteq STATES_A$ such that trunk$(\leftrightarrow_R^*) = L(A,Q)$.*

It is well known that for any tree automata A and B over a ranked alphabet Σ, and for any state sets $Q_A \subseteq STATES_A$ and $Q_B \subseteq STATES_B$, one can effectively decide if $L(A, Q_A) = L(A, Q_B)$. Hence by Proposition 2.7, one can effectively decide for given gtrs's R and S over Σ if $trunk(\leftrightarrow_R^*) = trunk(\leftrightarrow_S^*)$.

For further results on tree automata, see [9].

2.1. Ground term rewrite systems and tree automata

In this section we briefly recall some results on the connections between ground term rewrite systems and tree automata. First we adopt the main concepts and results of the paper [5]. Let E be a reduced gtrs over a ranked alphabet Σ. Let

$$\Theta = \overset{*}{\underset{E}{\leftrightarrow}} \cap (sub(E) \times sub(E)).$$

It should be clear that Θ is an equivalence relation on $sub(E)$. Furthermore, for any $t \in sub(E)$, we have $[t]_\Theta = [t]_{\leftrightarrow_E^*} \cap sub(E)$. Note that $sub(E)$ is finite. It is well known that for any terms $p, t \in T_\Sigma$, it is decidable if $p \leftrightarrow_E^* t$, see Proposition 2.3. Hence we can effectively construct Θ. Let $STATES = \{[t]_\Theta \mid t \in sub(E)\}$. We define the ranked alphabet $\Sigma \cup STATES$, where the elements of $STATES$ are viewed as symbols with rank 0. The tree automaton A over Σ is introduced as follows. Let A be the set of rules of the form

$$f([t_1]_\Theta, \ldots, [t_m]_\Theta) \rightarrow [f(t_1, \ldots, t_m)]_\Theta.$$

where $f \in \Sigma_m$, $m \geqslant 0$, $t_1, \ldots, t_m \in T_\Sigma$, and $f(t_1, \ldots, t_m) \in sub(E)$. Note that $f(t_1, \ldots, t_m) \in sub(E)$ implies that $t_i \in sub(E)$ for $1 \leqslant i \leqslant m$. By direct inspection of the tree automaton A we get that A is deterministic.

We now recall some important properties of A.

Proposition 2.8 (Fülöp and Vágvölgyi [5]). *A is a reduced gtrs over the ranked alphabet $\Sigma \cup STATES$.*

Proposition 2.9 (Fülöp and Vágvölgyi [5]). *For every $t \in sub(E)$, we have $t \rightarrow_A^* [t]_\Theta$.*

Proposition 2.10 (Fülöp and Vágvölgyi [5]). *For every $p \in T_\Sigma$ and $t \in sub(E)$, if $p \rightarrow_A^* [t]_\Theta$, then $p \leftrightarrow_E^* t$.*

Proposition 2.11 (Fülöp and Vágvölgyi [5]). $\leftrightarrow_E^* = \leftrightarrow_A^* \cap T_\Sigma \times T_\Sigma$.

We now show another property of $\leftrightarrow_A^*$ which is a generalization of Proposition 2.9.

Lemma 2.12. *For every $t \in sub(E)$, $L(A, [t]_\Theta) = [t]_{\leftrightarrow_E^*}$.*

Proof. By Propositions 2.9 and 2.11. □

Lemma 2.13. *For every $t \in trunk(\leftrightarrow_E^*)$, there is a tree $u \in sub(E)$ such that $t \rightarrow_A^* [u]_\Theta$.*

Proof. Let us assume that $t \in trunk(\leftrightarrow_E^*)$. Then by Proposition 2.2, $t \leftrightarrow_E^* u$ for some $u \in sub(E)$. By Proposition 2.11, $t \leftrightarrow_A^* u$. By Proposition 2.9, $u \rightarrow_A^* [u]_\Theta$. Thus $t \leftrightarrow_A^* [u]_\Theta$. The state $[u]_\Theta$ is irreducible for A. Hence by Propositions 2.8 and 2.1, $t \rightarrow_A^* [u]_\Theta$. □

Lemma 2.14. $trunk(\leftrightarrow_E^*) = L(A, STATES)$.

Proof. By Proposition 2.2 and Proposition 2.10, $L(A, STATES) \subseteq trunk(\leftrightarrow_E^*)$. By Lemma 2.13, $trunk(\leftrightarrow_E^*) \subseteq L(A, STATES)$. □

In the light of Proposition 2.3, we sum up the results of the section.

Theorem 2.15. *Let R be an arbitrary gtrs over a ranked alphabet Σ. Then we can effectively construct a deterministic tree automaton A over Σ such that*
(i) $\leftrightarrow_R^* = \leftrightarrow_A^* \cap T_\Sigma \times T_\Sigma$,
(ii) $trunk(\leftrightarrow_R^*) = L(A, STATES)$, *and*
(iii) *for each state $a \in STATES$, $L(A, a)$ is a congruence class of $\leftrightarrow_R^*$.*

Let R be an arbitrary gtrs over Σ and A be any deterministic tree automaton over Σ. Then (ii) and (iii) if and only if $stub(\leftrightarrow_R^*) = \{L(A, a) \mid a \in STATES\}$. Hence we can write Theorem 2.15 in the following form.

Theorem 2.16. *Let R be an arbitrary gtrs over a ranked alphabet Σ. Then we can effectively construct a deterministic tree automaton A over Σ such that*
(i) $\leftrightarrow_R^* = \leftrightarrow_A^* \cap T_\Sigma \times T_\Sigma$ *and*
(ii) $stub(\leftrightarrow_R^*) = \{L(A, a) \mid a \in STATES\}$.

Let A be a deterministic tree automaton over Σ. The ground tree transduction π_A induced by A is defined as follows. For any trees $p, q \in T_\Sigma$, $(p, q) \in \pi_A$ if and only if $p \rightarrow_A^* u$ and $q \rightarrow_A^* u$ for some $u \in T_{\Sigma \cup STATES}$. By Propositions 2.1 and 2.5, A is a convergent gtrs over the ranked alphabet $\Sigma \cup STATES$. Hence $\pi_A = \leftrightarrow_A^* \cap T_\Sigma \times T_\Sigma$. In the light of this observation we recall Theorem 3.3 in [7].

Proposition 2.17 (Fülöp and Vágvölgyi [7]). *For a given deterministic tree automaton A over Σ, we can effectively construct a reduced gtrs R over Σ such that $\leftrightarrow_R^* = \leftrightarrow_A^* \cap T_\Sigma \times T_\Sigma$.*

2.2. Tree language $\rho \diamond \tau$

Let ρ and τ be arbitrary congruences on **TA**. We define the ranked alphabet $\Gamma^{\rho,\tau}$ as follows. Let $\Gamma^{\rho,\tau} = \Gamma_0^{\rho,\tau}$, and $\Gamma_0^{\rho,\tau} \subseteq stub(\rho) \times stub(\rho) \times stub(\tau)$, and let $\Gamma_0^{\rho,\tau}$ consist of all triples $\lceil Z_1, Z_2, Z_3 \rceil$ where $Z_1, Z_2 \in stub(\rho)$, $Z_3 \in stub(\tau)$, $Z_1 \cap Z_3 \neq \emptyset$, and $Z_2 \cap Z_3 \neq \emptyset$. If ρ and τ are understood from the context, then we simply write Γ for $\Gamma^{\rho,\tau}$. We define the alphabet $\Delta^{\rho,\tau}$ by $\Delta^{\rho,\tau} = \Sigma \cup \Gamma$. From now on if ρ and τ are understood from the context, then we simply write Δ for $\Delta^{\rho,\tau}$.

Let $t \in T_\Delta$ be arbitrary.

(i) We define tree $t_{fi} \in T_\Delta(stub(\rho))$ from t by replacing each leaf $\lceil Z_1, Z_2, Z_3 \rceil$ by its first component Z_1.

(ii) We define tree $t_{se} \in T_\Delta(stub(\rho))$ from t by replacing each leaf $\lceil Z_1, Z_2, Z_3 \rceil$ by its second component Z_2.

(iii) We define tree $t_{th} \in T_\Delta(stub(\tau))$ from t by replacing each leaf $\lceil Z_1, Z_2, Z_3 \rceil$ by its third component Z_3.

We now introduce the tree language $\rho \diamond \tau$ over Δ, which we call the *cooperation between* ρ and τ *in inducing* $\rho \cap \tau$ *in* $trunk(\rho)$ *beyond* $trunk(\tau)$. Tree language $\rho \diamond \tau$ consists of all trees $t \in T_\Delta$ such that Conditions (a) and (b) hold.

(a) — $t_{fi}^{\mathbf{TA}/\rho} = t_{se}^{\mathbf{TA}/\rho} = Z$ for some ρ-class Z,
— $t_{fi} = f(t_{11}, \ldots, t_{1m})$, $f \in \Delta_m$, $m \geqslant 1$,
— for each i, $1 \leqslant i \leqslant m$, $t_{1i}^{\mathbf{TA}/\rho} = Z_{1i}$ for some ρ-class Z_{1i},
— $f^{\mathbf{TA}/\rho}(Z_{11}, \ldots, Z_{1m}) = Z$,
— $t_{se} = f(t_{21}, \ldots, t_{2m})$,
— for each i, $1 \leqslant i \leqslant m$, $t_{2i}^{\mathbf{TA}/\rho} = Z_{2i}$ for some ρ-class Z_{2i},
— $f^{\mathbf{TA}/\rho}(Z_{21}, \ldots, Z_{2m}) = Z$, and
— there is an integer j, $1 \leqslant j \leqslant m$, such that $Z_{1j} \neq Z_{2j}$.

(b) For any $g \in \Delta_n$, $n \geqslant 0$, $Z_1, \ldots, Z_n \in stub(\tau)$, if $g(Z_1, \ldots, Z_n)$ is a subtree of t_{th}, then $g^{\mathbf{TA}/\tau}(Z_1, \ldots, Z_n) \notin stub(\tau)$.

Note that Condition (a) implies that $Z \in comp(\rho)$.

2.3. Examples

We now illustrate the above concepts by three examples.

First example. Vágvölgyi [19] gave a ranked alphabet Σ and ground term rewrite systems R and S over Σ such that there is no gtrs U over Σ with $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$. We now adopt this example from [19]. Let $\Sigma = \{f, g, \#\}$ be a ranked alphabet, where f and g are of rank 1, and # is of rank 0. In this example we simply write strings for trees over Σ. For example, we write $fgf\sharp$ for $f(g(f(\sharp)))$. Furthermore, we also write $(fg)^2\sharp$ for $fgfg\sharp$, and $(fg)^3\sharp$ for $fgfgfg\sharp$, and so on.

Let the gtrs R consist of the following three rules:

$$ff\# \to f\#,$$

$$fg\# \to \#,$$

$$gf\# \to \#.$$

Let the gtrs S consist of the rule

$$g\# \to \#.$$

Proposition 2.18 (Vágvölgyi [19]). *There is no gtrs U over Σ such that $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$.*

Proof. By contradiction. Assume that there is a gtrs U over Σ such that

$$\overset{*}{\underset{U}{\leftrightarrow}} = \overset{*}{\underset{R}{\leftrightarrow}} \cap \overset{*}{\underset{S}{\leftrightarrow}}. \tag{1}$$

We obtain by direct inspection that for each $k \geqslant 1$,

$$f^2(gf)^k\# \overset{*}{\underset{R}{\rightarrow}} f\#$$

and

$$f^2(gf)^k g\# = f(fg)^{k+1}\# \overset{*}{\underset{R}{\rightarrow}} f\#.$$

Hence

$$f^2(gf)^k\# \overset{*}{\underset{R}{\leftrightarrow}} f^2(gf)^k g\# \quad \text{for } k \geqslant 1.$$

We obtain by direct inspection that

$$f^2(gf)^k\# \underset{S}{\leftrightarrow} f^2(gf)^k g\# \quad \text{for } k \geqslant 1.$$

Hence

$$(f^2(gf)^k\#, f^2(gf)^k g\#) \in \overset{*}{\underset{R}{\leftrightarrow}} \cap \overset{*}{\underset{S}{\leftrightarrow}} \quad \text{for } k \geqslant 1. \tag{2}$$

However,

$$(f(gf)^k\#, f(gf)^k g\#) \notin \overset{*}{\underset{R}{\leftrightarrow}} \quad \text{for } k \geqslant 1.$$

Hence

$$(f(gf)^k\#, f(gf)^k g\#) \notin \overset{*}{\underset{R}{\leftrightarrow}} \cap \overset{*}{\underset{S}{\leftrightarrow}} \quad \text{for } k \geqslant 1. \tag{3}$$

By (2) and (3), for each $k \geqslant 1$, $[f^2(gf)^k\#]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}$ is a compound $\leftrightarrow_R^* \cap \leftrightarrow_S^*$-class. By (1) and Proposition 2.2, $trunk(\leftrightarrow_R^* \cap \leftrightarrow_S^*)$ is the union of finitely many $\leftrightarrow_R^* \cap \leftrightarrow_S^*$-classes. Thus there are integers $1 \leqslant i < j$ such that the trees $f^2(gf)^i\#$ and $f^2(gf)^j\#$ are in the same $\leftrightarrow_R^* \cap \leftrightarrow_S^*$-class. That is,

$$[f^2(gf)^i\#]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*} = [f^2(gf)^j\#]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}. \tag{4}$$

On the other hand, we obtain by direct inspection that the trees $f^2(gf)^k\#$, $k \geqslant 1$, are irreducible for S. As S is reduced, S is convergent. Hence the trees $f^2(gf)^l\#$ and $f^2(gf)^m\#$, $1 \leqslant l < m$, are in different $\leftrightarrow_S^*$-classes. Hence the trees $f^2(gf)^l\#$ and $f^2(gf)^m\#$, $1 \leqslant l < m$, are in different $\leftrightarrow_R^* \cap \leftrightarrow_S^*$-classes. By (4), this is a contradiction. □

We now compute ranked alphabets $\Gamma^{\rho,\tau}$, $\Delta^{\rho,\tau}$ and tree language $\rho \diamond \tau$ over $\Delta^{\rho,\tau}$. We obtain by direct inspection that

$$stub\left(\overset{*}{\underset{R}{\leftrightarrow}}\right) = \{[\#]_{\leftrightarrow_R^*}, [f(\#)]_{\leftrightarrow_R^*}, [g(\#)]_{\leftrightarrow_R^*}\}$$

and $f^{\mathbf{TA}}([\#]_{\leftrightarrow_R^*})=[f(\#)]_{\leftrightarrow_R^*}$, $f^{\mathbf{TA}}([f(\#)]_{\leftrightarrow_R^*})=[f(\#)]_{\leftrightarrow_R^*}$, $g^{\mathbf{TA}}([\#]_{\leftrightarrow_R^*})=[g(\#)]_{\leftrightarrow_R^*}$, $f^{\mathbf{TA}}([g(\#)]_{\leftrightarrow_R^*})=[\#]_{\leftrightarrow_R^*}$, $g^{\mathbf{TA}}([f(\#)]_{\leftrightarrow_R^*})=[\#]_{\leftrightarrow_R^*}$.

Furthermore,

$$stub(\underset{S}{\overset{*}{\leftrightarrow}}) = \{[\#]_{\leftrightarrow_S^*}\}$$

and $g^{\mathbf{TA}}([\#]_{\leftrightarrow_S^*})=[\#]_{\leftrightarrow_S^*}$.

Observe that $[\#]_{\leftrightarrow_R^*}\cap[\#]_{\leftrightarrow_S^*}\neq\emptyset$, $[f(\#)]_{\leftrightarrow_R^*}\cap[\#]_{\leftrightarrow_S^*}=\emptyset$, $[g(\#)]_{\leftrightarrow_R^*}\cap[\#]_{\leftrightarrow_S^*}\neq\emptyset$. Hence ranked alphabet Γ consists of four nullary symbols.

$$\Gamma = \{\lceil [\#]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil, \lceil [\#]_{\leftrightarrow_R^*}, [g(\#)]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil,$$
$$\lceil [g(\#)]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil, \lceil [g(\#)]_{\leftrightarrow_R^*}, [g(\#)]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil\}.$$

Then $\Delta=\Delta_0\cup\Delta_1$, $\Delta_1=\{f,g\}$ and $\Delta_0=\Gamma$.

Let $k\geqslant 1$ be arbitrary. Let

$$t = f^2(gf)^k\lceil [\#]_{\leftrightarrow_R^*}, [g(\#)]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil.$$

By the proof of Proposition 2.18, Conditions (a) and (b) hold. Hence

$$\text{for each } k \geqslant 1, \quad f^2(gf)^k\lceil [\#]_{\leftrightarrow_R^*}, [g(\#)]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil \in \rho \diamond \tau.$$

Hence tree language $\rho\diamond\tau$ is infinite.

Second example. Let $\Sigma=\{f,\#,\$\}$ be a ranked alphabet, where f is of rank 1, $\#$ and $\$$ are of rank 0. Let the gtrs R consist of the following rule:

$$f(\#) \to f(\$).$$

Let the gtrs S consist of the rule

$$\# \to \$.$$

We obtain by direct inspection that

- $stub(\leftrightarrow_R^*)=\{[\#]_{\leftrightarrow_R^*}, [\$]_{\leftrightarrow_R^*}, [f(\#)]_{\leftrightarrow_R^*}\}$,
- $f^{\mathbf{TA}}([\#]_{\leftrightarrow_R^*})=[f(\#)]_{\leftrightarrow_R^*}$, $f^{\mathbf{TA}}([\$]_{\leftrightarrow_R^*})=[f(\#)]_{\leftrightarrow_R^*}$,
- $stub(\leftrightarrow_S^*)=\{[\#]_{\leftrightarrow_S^*}\}$,
- $[\#]_{\leftrightarrow_R^*}\cap[\#]_{\leftrightarrow_S^*}\neq\emptyset$, and $[\$]_{\leftrightarrow_R^*}\cap[\#]_{\leftrightarrow_S^*}\neq\emptyset$.

We carry out the following two symmetrical steps. In Step 1 we compute ranked alphabets $\Gamma^{\leftrightarrow_R^*,\leftrightarrow_S^*}$, $\Delta^{\leftrightarrow_R^*,\leftrightarrow_S^*}$ and tree language $\rho\diamond\tau$ over $\Delta^{\leftrightarrow_R^*,\leftrightarrow_S^*}$. In Step 2 we compute ranked alphabets $\Gamma^{\leftrightarrow_S^*,\leftrightarrow_R^*}$, $\Delta^{\leftrightarrow_S^*,\leftrightarrow_R^*}$ and tree language $\tau\diamond\rho$ over $\Delta^{\leftrightarrow_S^*,\leftrightarrow_R^*}$.

Step 1. We now simply write Γ for $\Gamma^{\leftrightarrow_R^*,\leftrightarrow_S^*}$ and Δ for $\Delta^{\leftrightarrow_R^*,\leftrightarrow_S^*}$. Ranked alphabet Γ consists of four nullary symbols.

$$\Gamma = \{\lceil [\#]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil, \lceil [\#]_{\leftrightarrow_R^*}, [\$]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil, \lceil [\$]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil,$$
$$\lceil [\$]_{\leftrightarrow_R^*}, [\$]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*}\rceil\}.$$

$\Delta = \Delta_0 \cup \Delta_1$, $\Delta_0 = \Gamma$, and $\Delta_1 = \{f\}$. We obtain by direct inspection that tree language $\rho \diamond \tau$ consists of the only tree $f(\lceil [\#]_{\leftrightarrow_R^*}, [\$]_{\leftrightarrow_R^*}, [\#]_{\leftrightarrow_S^*} \rceil)$.

Step 2. We now simply write Γ for $\Gamma^{\leftrightarrow_S^*, \leftrightarrow_R^*}$ and Δ for $\Delta^{\leftrightarrow_S^*, \leftrightarrow_R^*}$. We obtain by direct inspection that ranked alphabet Γ consists of two nullary symbols.

$$\Gamma = \{\lceil [\#]_{\leftrightarrow_S^*}, [\#]_{\leftrightarrow_S^*}, [\#]_{\leftrightarrow_R^*} \rceil, \lceil [\#]_{\leftrightarrow_S^*}, [\#]_{\leftrightarrow_S^*}, [\$]_{\leftrightarrow_R^*} \rceil\}.$$

$\Delta = \Delta_0 \cup \Delta_1$, $\Delta_0 = \Gamma$, and $\Delta_1 = \{f\}$. We obtain by direct inspection that $\tau \diamond \rho = \emptyset$.

Thus both $\rho \diamond \tau$ and $\tau \diamond \rho$ are finite. By Statement 1.5, we can effectively construct a gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$. In fact as $\leftrightarrow_R^* \subseteq \leftrightarrow_S^*$, $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_S^*$.

We now present our third example.

Third example. Let $\Sigma = \{f, g, h, \#\}$ be a ranked alphabet, where f, g, and h are of rank 1, and # is of rank 0. Let the gtrs R consist of the following four rules:

$$f(f(\#)) \to \#,$$

$$f(g(\#)) \to \#,$$

$$g(f(\#)) \to \#,$$

$$g(g(\#)) \to \#.$$

Let the gtrs S consist of the following four rules:

$$h(h(\#)) \to \#,$$

$$h(g(\#)) \to \#,$$

$$g(h(\#)) \to \#,$$

$$g(g(\#)) \to \#.$$

Let the gtrs U consist of the only rule:

$$g(g(\#)) \to \#.$$

We obtain by direct inspection that

- $stub(\leftrightarrow_R^*) = \{[\#]_{\leftrightarrow_R^*}, [f(\#)]_{\leftrightarrow_R^*}, [g(\#)]_{\leftrightarrow_R^*}\}$,
- $stub(\leftrightarrow_S^*) = \{[\#]_{\leftrightarrow_S^*}, [h(\#)]_{\leftrightarrow_S^*}, [g(\#)]_{\leftrightarrow_S^*}\}$,
- $\rho \diamond \tau = \emptyset$,
- $\tau \diamond \rho = \emptyset$, and
- $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$.

2.4. Basic notations

From now on, throughout the paper let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. By Theorem 2.16, we can effectively construct deterministic tree automata A and B over Σ such that

(I) $\leftrightarrow_R^* = \leftrightarrow_A^* \cap T_\Sigma \times T_\Sigma$,
(II) $stub(\leftrightarrow_R^*) = \{L(A,a) \mid a \in STATES_A\}$,
(III) $\leftrightarrow_S^* = \leftrightarrow_B^* \cap T_\Sigma \times T_\Sigma$, and
(IV) $stub(\leftrightarrow_S^*) = \{L(B,b) \mid b \in STATES_B\}$.

Recall that $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is a tree language over the ranked alphabet $\Delta^{\leftrightarrow_R^*, \leftrightarrow_S^*} = \Sigma \cup \Gamma^{\leftrightarrow_R^*, \leftrightarrow_S^*}$. Symmetrically, $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ is a tree language over the ranked alphabet $\Delta^{\leftrightarrow_S^*, \leftrightarrow_R^*} = \Sigma \cup \Gamma^{\leftrightarrow_S^*, \leftrightarrow_R^*}$. From now on throughout this section, we write Γ for $\Gamma^{\leftrightarrow_R^*, \leftrightarrow_S^*}$ and Δ for $\Delta^{\leftrightarrow_R^*, \leftrightarrow_S^*}$. Recall the ranked alphabet $\Gamma = \Gamma_0$ consist of all triples $\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil$ of congruence classes where

- $a_1, a_2 \in STATES_A$, $b \in STATES_B$ and
- $L(A,a_1) \cap L(B,b) \neq \emptyset$ and $L(A,a_2) \cap L(B,b) \neq \emptyset$.

We now introduce five types of mappings. The Γ-evaluation of type 1 is a mapping $\psi_1 : T_\Delta \to [T_\Sigma]_{\leftrightarrow_R^*}$ such that the following conditions hold. For each $\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil \in \Gamma$, $\psi_1(\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil) = L(A,a_1)$. For any $f \in \Sigma_m$, $m \geqslant 0$, and $t_1, \ldots, t_m \in T_\Delta$,

$$\psi_1(f(t_1,\ldots,t_m)) = f^{\mathbf{TA}/\leftrightarrow_R^*}(\psi_1(t_1),\ldots,\psi_1(t_m)).$$

The Γ-evaluation of type 2 is a mapping $\psi_2 : T_\Delta \to [T_\Sigma]_{\leftrightarrow_S^*}$ such that the following conditions hold. For each $\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil \in \Gamma$,

$$\psi_2(\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil) = L(A,a_2).$$

For any $f \in \Sigma_m$, $m \geqslant 0$, and $t_1, \ldots, t_m \in T_\Delta$,

$$\psi_2(f(t_1,\ldots,t_m)) = f^{\mathbf{TA}/\leftrightarrow_R^*}(\psi_2(t_1),\ldots,\psi_2(t_m)).$$

The Γ-evaluation of type 3 is a mapping $\psi_3 : T_\Delta \to T_\Sigma(STATES_B)$ such that the following conditions hold. For each $\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil \in \Gamma$, $\psi_3(\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil) = b$. For any $f \in \Sigma_m$, $m \geqslant 0$, and $t_1, \ldots, t_m \in T_\Delta$,

$$\psi_3(f(t_1,\ldots,t_m)) = f(\psi_3(t_1),\ldots,\psi_3(t_m)).$$

A Γ-evaluation of type 4 is a mapping $\psi_4 : T_\Delta \to T_\Sigma$ such that the following conditions hold. For each $\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil \in \Gamma$,

$$\psi_4(\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil) \in L(A,a_1) \cap L(B,b).$$

For any $f \in \Sigma_m$, $m \geqslant 0$, and $t_1, \ldots, t_m \in T_\Delta$,

$$\psi_4(f(t_1,\ldots,t_m)) = f(\psi_4(t_1),\ldots,\psi_4(t_m)).$$

A Γ-evaluation of type 5 is a mapping $\psi_5 : T_\Delta \to T_\Sigma$ such that the following conditions hold. For each $\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil \in \Gamma$,

$$\psi_5(\lceil L(A,a_1), L(A,a_2), L(B,b) \rceil) \in L(A,a_2) \cap L(B,b).$$

For any $f \in \Sigma_m$, $m \geqslant 0$, and $t_1, \ldots, t_m \in T_\Delta$,

$$\psi_5(f(t_1, \ldots, t_m)) = f(\psi_5(t_1), \ldots, \psi_5(t_m)).$$

3. Tree language $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is recognizable

Recall that gtrs's R and S, deterministic tree automata A and B over Σ, and ranked alphabets $\Gamma^{\leftrightarrow_R^*, \leftrightarrow_S^*}$ and $\Delta^{\leftrightarrow_R^*, \leftrightarrow_S^*}$ were introduced in Section 2.4. From now on throughout this section, we write Γ for $\Gamma^{\leftrightarrow_R^*, \leftrightarrow_S^*}$ and Δ for $\Delta^{\leftrightarrow_R^*, \leftrightarrow_S^*}$.

We now construct the state set $STATES_C$ of the deterministic tree automaton C over Δ. Let $STATES_C = STATES_1 \cup STATES_2$, where

- $STATES_1 = STATES_A \times STATES_A \times STATES_B$. We denote the elements of $STATES_1$ by $\|a_1, a_2, b\|$, where $a_1, a_2 \in STATES_A$ and $b \in STATES_B$.
- $STATES_2 = STATES_A \times STATES_A \times \{*, \$\}$, where $*, \$ \notin STATES_B$. We denote the elements of $STATES_2$ by $\langle a_1, a_2, * \rangle$ and $\langle a_1, a_2, \$ \rangle$, respectively, where $a_1, a_2 \in STATES_A$.

We now construct the deterministic tree automaton C over Δ.

(1) For each $\lceil L(A, a_1), L(A, a_2), L(B, b) \rceil \in \Gamma$, we put the rule

$$\lceil L(A, a_1), L(A, a_2), L(B, b) \rceil \to \|a_1, a_2, b\|$$

in C.

(2) We put the rule $f(c_1, \ldots, c_m) \to c$ in C, where
  - (a) $f \in \Sigma_m$, $m \geqslant 1$,
  - (b) for each $1 \leqslant i \leqslant m$, $c_i \in STATES_C$,
  - (c) $c \in STATES_2$,
  - (d) rules $f(a_{11}, \ldots, a_{1m}) \to a_1$ and $f(a_{21}, \ldots, a_{2m}) \to a_2$ are in A,
  - (e) $c_i \in \{\langle a_{1i}, a_{2i}, * \rangle, \langle a_{1i}, a_{2i}, \$ \rangle \mid a_{1i}, a_{2i} \in STATES_A\}$ for some $1 \leqslant i \leqslant m$, or for each $1 \leqslant i \leqslant m$, $c_i = \|a_{1i}, a_{2i}, b_i\|$ for some $b_i \in B$ and there is no rule in B with left-hand side $f(b_1, \ldots, b_m)$,
  - (f) $c = \langle a_1, a_2, * \rangle$ and $a_1 = a_2$ if $a_{1i} = a_{2i}$ for $1 \leqslant i \leqslant m$, and
  - (g) $c = \langle a_1, a_2, \$ \rangle$ if $a_{1i} \neq a_{2i}$ for some $1 \leqslant i \leqslant m$.

We define the set $TEST$ of states as follows. Let

$$TEST = \{\langle a, a, \$ \rangle \mid a \in STATES_A\}.$$

Lemma 3.1. *Let $p \in T_\Delta$. Let $p \to_C^* \|a_1, a_2, b\|$ for some $a_1, a_2 \in A$ and $b \in B$. Then $p = \lceil L(A, a_1), L(A, a_2), L(B, b) \rceil \in \Gamma$.*

Proof. Observe that for any rule of C defined by Condition (2) in the definition of C, the third component of the right-hand side is $*$ or \$. Hence $p = \lceil L(A, a_1), L(A, a_2), L(B, b) \rceil \in \Gamma$ for some $a_1, a_2 \in A$ and $b \in B$, and the rule

$$\lceil L(A, a_1), L(A, a_2), L(B, b) \rceil \to \|a_1, a_2, b\|$$

is in C. $\square$

Lemma 3.2. *Let* $p \in T_\Delta$. *Let* $\psi_1 : T_\Delta \to [T_\Sigma]_{\leftrightarrow_R^*}$ *be the* Γ*-evaluation of type* 1, *and let* $\psi_2 : T_\Delta \to [T_\Sigma]_{\leftrightarrow_S^*}$ *be the* Γ*-evaluation of type* 2, *and let* $\psi_3 : T_\Delta \to T_\Sigma(STATES_B)$ *be the* Γ*-evaluation of type* 3. *Let* $a_1, a_2 \in STATES_A$. *Then*

$$\psi_1(p) = L(A, a_1),\ \psi_2(p) = L(A, a_2),\ \text{and } \psi_3(p) \text{ is irreducible for } B$$

if and only if

$$p \to_C^* c \quad \text{with } c \in \{\|a_1, a_2, b\|, \langle a_1, a_2, *\rangle, \langle a_1, a_2, \$\rangle \mid b \in STATES_B\}.$$

Proof. ($\Rightarrow$) We proceed by induction on *height*(p). *Base case*: Let *height*$(p) = 0$. Then $p \in \Delta_0$. As $\psi_1(p) = L(A, a_1)$, by the definition of ψ_1, $p \notin \Sigma_0$. Hence $p \in \Gamma$. Then $p = \lceil L(A, a_1), L(A, a_2), L(B, b)\rceil$ for some $a_1, a_2 \in A$ and $b \in B$. Hence by Condition (1) in the definition of C, the rule

$$\lceil L(A, a_1), L(A, a_2), L(B, b)\rceil \to \|a_1, a_2, b\|$$

is in C.

Induction step: Let $n \geqslant 1$. Assume that the statement is shown for all trees p of height less than n. Let *height*$(p) = n$. Then $p = f(p_1, \ldots, p_m)$ for some $f \in \Sigma_m$, $m \geqslant 1$, and $p_1, \ldots, p_m \in T_\Delta$. Furthermore, $\psi_1(p_i) = L(A, a_{1i})$ for some $a_{1i} \in A$ for $1 \leqslant i \leqslant m$ and $\psi_2(p_i) = L(A, a_{2i})$ for some $a_{2i} \in A$ for $1 \leqslant i \leqslant m$. Hence

$$L(A, a_1) = f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A, a_{11}), \ldots, L(A, a_{1m}))$$

and

$$L(A, a_2) = f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A, a_{21}), \ldots, L(A, a_{2m})).$$

By the definition of A, the rules

$$f(a_{11}, \ldots, a_{1m}) \to a_1$$

and

$$f(a_{21}, \ldots, a_{2m}) \to a_2$$

are in A.

Since $\psi_3(p)$ is irreducible for B, for each $1 \leqslant i \leqslant m$, $\psi_3(p_i)$ is irreducible for B. Hence by the induction hypothesis for each $1 \leqslant i \leqslant m$, $p_i \to_C^* c_i$, where $c_i \in \{\|a_{1i}, a_{2i}, b\|, \langle a_{1i}, a_{2i}, *\rangle, \langle a_{1i}, a_{2i}, \$\rangle \mid b \in STATES_B\}$. If $c_i \in \{\langle a_{1i}, a_{2i}, *\rangle, \langle a_{1i}, a_{2i}, \$\rangle \mid a_{1i}, a_{2i} \in STATES_A\}$ for some $1 \leqslant i \leqslant m$, then Condition 2(e) in the definition of C holds. Assume that for each $1 \leqslant i \leqslant m$, $c_i = \|a_{1i}, a_{2i}, b_i\|$ for some $b_i \in B$. Then by the definition of C, $p_i = \lceil L(A, a_1), L(A, a_2), L(B, b)\rceil \in \Gamma$ for $1 \leqslant i \leqslant m$, and hence

$$p = f(\lceil L(A, a_{11}), L(A, a_{21}), L(B, b_1)\rceil, \ldots, \lceil L(A, a_{1m}), L(A, a_{2m}), L(B, b_m)\rceil).$$

Hence $\psi_3(p)=f(b_1,\ldots,b_m)$. As $\psi_3(p)$ is irreducible, there is no rule in B with left-hand side $f(b_1,\ldots,b_m)$. Thus Condition 2(e) in the definition of C holds. Let $c=\langle a_1, a_2,*\rangle$ if $a_{1i}=a_{2i}$ for $1\leqslant i\leqslant m$, and $c=\langle a_1,a_2,\$\rangle$ if $a_{1i}\neq a_{2i}$ for some $1\leqslant i\leqslant m$. By Condition (2) in the definition of C, rule $f(c_1,\ldots,c_m)\to c$ is in C. Thus

$$f(p_1,\ldots,p_m)\xrightarrow[C]{*} f(c_1,\ldots,c_m)\to c.$$

($\Leftarrow$) We proceed by induction on *height*(p). *Base case*: Let *height*$(p)=0$. Then $p\in\Delta_0$. By the definition of C, $p=\lceil L(A,a_1),L(A,a_2),L(B,b)\rceil\in\Gamma$ for some $a_1,a_2\in A$ and $b\in B$, and the rule

$$\lceil L(A,a_1),L(A,a_2),L(B,b)\rceil\to\|a_1,a_2,b\|$$

is in C. Hence $\psi_1(p)=L(A,a_1)$, $\psi_2(p)=L(A,a_2)$, and $\psi_3(p)=b$ is irreducible for B.

Induction step: Let $n\geqslant 1$. Assume that the statement is shown for all trees p of height less than n. Let *height*$(p)=n$. Then $p=f(p_1,\ldots,p_m)$ for some $f\in\Sigma_m$, $m\geqslant 1$, and $p_1,\ldots,p_m\in T_\Sigma$. Thus

$$f(p_1,\ldots,p_m)\xrightarrow[C]{*} f(c_1,\ldots,c_m)\to c.$$

For each $1\leqslant i\leqslant m$, $p_i\to_C^* c_i$, where $c_i\in\{\|a_{1i},a_{2i},b\|,\langle a_{1i},a_{2i},*\rangle,\langle a_{1i},a_{2i},\$\rangle\mid b\in STATES_B\}$ for some $a_{1i},a_{2i}\in A$. The rule $f(c_1,\ldots,c_m)\to c$ is in C. By Condition (2) in the definition of C the rules

$$f(a_{11},\ldots,a_{1m})\to a_1,$$

$$f(a_{21},\ldots,a_{2m})\to a_2$$

are in A. Furthermore, if for each $1\leqslant i\leqslant m$, $c_i=\|a_{1i},a_{2i},b_i\|$ for some $b_i\in B$, then there is no rule in B with left-hand side $f(b_1,\ldots,b_m)$. Hence

$$L(A,a_1)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{11}),\ldots,L(A,a_{1m}))$$

and

$$L(A,a_2)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{21}),\ldots,L(A,a_{2m})).$$

By the induction hypothesis, $\psi_1(p_{1i})=L(A,a_{1i})$, $\psi_2(p_{2i})=L(A,a_{2i})$ for $1\leqslant i\leqslant m$, and $\psi_3(p_i)$ is irreducible for B for $1\leqslant i\leqslant m$. Hence $\psi_1(p)=L(A,a_1)$, $\psi_2(p)=L(A,a_2)$, and $\psi_3(p)$ is irreducible for B. $\square$

Lemma 3.3. *Let $t\in L(C,STATES_C)$ be arbitrary. Let $\psi_4:T_\Delta\to T_\Sigma$ be a Γ-evaluation of type* 4. *Then $\psi_4(t)$'s $\to_B$-normal form is $\psi_3(t)$.*

Proof. We proceed by induction on *height*(t). *Base case*: Let *height*$(t)=0$. Then $t\in\Delta_0$. By the definition of C, $t=\lceil L(A,a_1),L(A,a_2),L(B,b)\rceil\in\Gamma$ for some $a_1,a_2\in A$ and $b\in B$, and the rule

$$\lceil L(A,a_1),L(A,a_2),L(B,b)\rceil\to\|a_1,a_2,b\|$$

is in C. By the definition of a Γ-evaluation of type 4,

$$\psi_4(\lceil L(A,a_1), L(A,a_2), L(B,b)\rceil) \in L(A,a_1) \cap L(B,b).$$

Hence $\psi_4(t)$'s $\to_B$-normal form is $b=\psi_3(t)$.

Induction step: Let $n \geqslant 1$. Assume that the statement is shown for all trees t of height less than n. Let $height(t)=n$. Then $t=f(t_1,\ldots,t_m)$ for some $f \in \Sigma_m$, $m \geqslant 1$, and $t_1,\ldots,t_m \in T_\Sigma$. Thus

$$f(t_1,\ldots,t_m) \xrightarrow[C]{*} f(c_1,\ldots,c_m) \to c.$$

For each $1 \leqslant i \leqslant m$, $t_i \to_C^* c_i$, where $c_i \in L(C, STATES_C)$. By Lemma 3.2, $\psi_3(t)$ is irreducible for B. By the definition of the Γ-evaluation of type 3,

$$\psi_3(f(t_1,\ldots,t_m)) = f(\psi_3(t_1),\ldots,\psi_3(t_m)).$$

By the definition of a Γ-evaluation of type 4,

$$\psi_4(f(t_1,\ldots,t_m)) = f(\psi_4(t_1),\ldots,\psi_4(t_m)).$$

By the induction hypothesis, $\psi_4(t_i)$'s $\to_B$-normal form is $\psi_3(t_i)$ for $1 \leqslant i \leqslant m$. Hence $\psi_4(t)$'s $\to_B$-normal form is $\psi_3(t)$. $\square$

Lemma 3.4. *Let* $t \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$. *Let* $\psi_4 : T_\Delta \to T_\Sigma$ *be a* Γ*-evaluation of type* 4. *Then* $[\psi_4(t)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}$ *is a compound* $\leftrightarrow_R^* \cap \leftrightarrow_S^*$*-class.*

Proof. Since $t \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$, $t=f(t_1,\ldots,t_m)$ for some $f \in \Delta_m$, $m \geqslant 1$, and $t_1,\ldots,t_m \in T_\Delta$. Moreover, Conditions (a) and (b) in Section 2.2 hold for relations $\rho = \leftrightarrow_R^*$ and $\tau = \leftrightarrow_S^*$. By the definition of a Γ-evaluation of type 4, $\psi_4(t)=f(\psi_4(t_1),\ldots,\psi_4(t_m))$. Let $\psi_5 : T_\Delta \to T_\Sigma$ be a Γ-evaluation of type 5. Then by the definition of a Γ-evaluation of type 5, $\psi_5(t)=f(\psi_5(t_1),\ldots,\psi_5(t_m))$. Since $t \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$, $\psi_4(t) \leftrightarrow_R^* \psi_5(t)$. By the definitions of mappings ψ_4 and ψ_5, $\psi_4(t) \leftrightarrow_S^* \psi_5(t)$. Hence $[\psi_4(t)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*} = [\psi_5(t)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}$.

Thus

$$\begin{aligned} & f^{\leftrightarrow_R^* \cap \leftrightarrow_S^*}([\psi_4(t_1)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*},\ldots,[\psi_4(t_m)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}) \\ & \quad = [\psi_4(t)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*} = [\psi_5(t)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*} \\ & \quad = f^{\leftrightarrow_R^* \cap \leftrightarrow_S^*}([\psi_5(t_1)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*},\ldots,[\psi_5(t_m)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}). \end{aligned}$$

We obtain by direct inspection that for each $1 \leqslant i \leqslant m$, $[\psi_4(t_i)]_{\leftrightarrow_R^*} = Z_{1i}$ and $[\psi_5(t_i)]_{\leftrightarrow_R^*} = Z_{2i}$. By Condition (a) in Section 2.2 there is an integer j, $1 \leqslant j \leqslant m$, such that $Z_{1j} \neq Z_{2j}$. Hence $[\psi_4(t_j)]_{\leftrightarrow_R^*} \neq [\psi_5(t_j)]_{\leftrightarrow_R^*}$. Thus $[\psi_4(t_j)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*} \neq [\psi_5(t_j)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}$. Hence $[\psi_4(t)]_{\leftrightarrow_R^* \cap \leftrightarrow_S^*}$ is a compound $\leftrightarrow_R^* \cap \leftrightarrow_S^*$-class. $\square$

Lemma 3.5. $L(C, TEST) \subseteq \leftrightarrow_R^* \diamond \leftrightarrow_S^*$.

Proof. Let $p\in L(C, TEST)$ be arbitrary. Then $p\to_C^* c$ for some $c\in TEST$. Hence $c=\langle a,a,\$\rangle$. We obtain by direct inspection of the definition of C that $p=f(p_1,\ldots,p_m)$ for some $f\in\Sigma_m$, $m\geqslant 1$, and $p_1,\ldots,p_m\in T_\Delta$. Hence $p=f(p_1,\ldots,p_m)\to_C^* f(c_1,\ldots,c_m)\to_C c$, where the rule $f(c_1,\ldots,c_m)\to c$ is in C. Hence Conditions (b), (d), and (e) in (2) hold and there is an integer i, $1\leqslant i\leqslant m$, such that

$$a_{1i}\neq a_{2i}. \tag{5}$$

By Lemma 3.2
- $\psi_1(p_i)=L(A,a_{1i})$ for $1\leqslant i\leqslant m$,
- $\psi_2(p_i)=L(A,a_{2i})$ for $1\leqslant i\leqslant m$,
- $\psi_1(p)=L(A,a)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{11}),\ldots,L(A,a_{1m}))$,
- $\psi_2(p)=L(A,a)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{21}),\ldots,L(A,a_{2m}))$, and
- $\psi_3(p)$ is irreducible for B.

Hence
$\psi_1(p)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{11}),\ldots,L(A,a_{1m}))$ and
$\psi_2(p)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{21}),\ldots,L(A,a_{2m}))$.
As A is deterministic by (5)

$$L(A,a_{1i})\neq L(A,a_{2i}).$$

This implies that $p\in\leftrightarrow_R^*\diamond\leftrightarrow_S^*$. $\square$

Lemma 3.6. $\leftrightarrow_R^*\diamond\leftrightarrow_S^*\subseteq L(C, TEST)$.

Proof. Let $p\in\leftrightarrow_R^*\diamond\leftrightarrow_S^*$ be arbitrary. Then
- $p=f(p_1,\ldots,p_m)$, for some $f\in\Sigma_m$, $m\geqslant 1$, and $p_1,\ldots,p_m\in T_\Delta$.
- $\psi_1(p_i)=L(A,a_{1i})$ for some $a_{1i}\in STATES_A$ for $1\leqslant i\leqslant m$,
- $\psi_2(p_i)=L(A,a_{2i})$ for some $a_{2i}\in STATES_A$ for $1\leqslant i\leqslant m$,
- $\psi_1(p)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{11}),\ldots,L(A,a_{1m}))=L(A,a)$ for some $a\in STATES_A$,
- $\psi_2(p)=f^{\mathbf{TA}/\leftrightarrow_R^*}(L(A,a_{21}),\ldots,L(A,a_{2m}))=L(A,a)$,
- rules $f(a_{11},\ldots,a_{1m})\to a$ and $f(a_{21},\ldots,a_{2m})\to a$ are in A,
- there is an integer i, $1\leqslant i\leqslant m$, such that $a_{1i}\neq a_{2i}$,
- $\psi_3(p)$ is irreducible for B.

By Lemma 3.2 $p=f(p_1,\ldots,p_m)\to_C^* f(c_1,\ldots,c_m)\to_C c$, where
- for each $1\leqslant i\leqslant m$, $c_i\in\{\|a_{1i},a_{2i},b_i\|,\langle a_{1i},a_{2i},*\rangle,\langle a_{1i},a_{2i},\$\rangle \mid b_i\in STATES_B\}$,
- $c\in\{\|a,a,b\|,\langle a,a,*\rangle,\langle a,a,\$\rangle \mid b\in STATES_B\}$, and
- the rule $f(c_1,\ldots,c_m)\to c$ is in C.

Since $f\in\Sigma_m$ with $m\geqslant 1$, $c\in\{\langle a,a,*\rangle,\langle a,a,\$\rangle\}$. Since there is an integer i, $1\leqslant i\leqslant m$, such that $a_{1i}\neq a_{2i}$, we have $c=\langle a,a,\$\rangle$. Thus $p\in L(C)$. $\square$

Lemmas 3.5 and 3.6 imply the following result.

Theorem 3.7. $L(C, TEST)=\leftrightarrow_R^*\diamond\leftrightarrow_S^*$.

The following result is a consequence of Proposition 2.6 and Theorem 3.7.

Theorem 3.8. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. Then it is decidable if $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is finite.*

4. Case that tree language $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is infinite

In this section we study the case when $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is infinite. From now on throughout this section, we write Γ for $\Gamma^{\leftrightarrow_R^*, \leftrightarrow_S^*}$ and Δ for $\Delta^{\leftrightarrow_R^*, \leftrightarrow_S^*}$.

Theorem 4.1. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. If $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is infinite, then there is no gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$.*

Proof. By contradiction. Assume that $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is infinite, and that there is a gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$. By Theorem 2.16, there is a deterministic tree automaton D over Σ such that

- $\leftrightarrow_U^* = \leftrightarrow_D^* \cap T_\Sigma \times T_\Sigma$, and
- $stub(\leftrightarrow_U^*) = \{L(D,d) \mid d \in STATES_D\}$.

Hence

$$\overset{*}{\underset{R}{\leftrightarrow}} \cap \overset{*}{\underset{S}{\leftrightarrow}} = \overset{*}{\underset{D}{\leftrightarrow}} \cap T_\Sigma \times T_\Sigma. \tag{6}$$

Let k be the cardinality of $STATES_D$.

We constructed the deterministic tree automaton C over Δ in Section 3. By Theorem 3.7, $L(C, TEST) = \leftrightarrow_R^* \diamond \leftrightarrow_S^*$. Let $p \in L(C, TEST)$ such that $height(p) > k$. Then there is a state $c \in TEST$ such that $p \to_C^* c$. By the definition of $TEST$,

$$c = \langle a, a, \$\rangle \tag{7}$$

for some $a \in STATES_A$.

Let $\psi_4 : T_\Delta \to T_\Sigma$ be a Γ-evaluation of type 4. By Lemma 3.4, $\psi_4(p) \in trunk(\leftrightarrow_U^*)$. Hence $\psi_4(p) \to_D^* d_1$ for some state $d_1 \in STATES_D$. Since $height(p) > k$ and k is the cardinality of $STATES_D$, there are words $\alpha, \beta \in N^*$ and state $d_2 \in STATES_D$ such that $\alpha, \alpha\beta \in O(p)$, $\beta \neq \lambda$, and that

$$\psi_4(p)/\alpha \overset{*}{\underset{D}{\rightarrow}} d_2 \quad \text{and} \quad \psi_4(p)/\alpha\beta \overset{*}{\underset{D}{\rightarrow}} d_2.$$

Hence

$$\psi_4(p)[\alpha \leftarrow \psi_4(p)/\alpha\beta] \overset{*}{\underset{D}{\rightarrow}} d_1$$

as well. Thus

$$\psi_4(p) \overset{*}{\underset{D}{\leftrightarrow}} \psi_4(p)[\alpha \leftarrow \psi_4(p)/\alpha\beta].$$

We obtain by direct inspection that

$$\psi_4(p)[\alpha \leftarrow \psi_4(p)/\alpha\beta] = \psi_4(p[\alpha \leftarrow p/\alpha\beta]).$$

Thus

$$\psi_4(p) \overset{*}{\underset{D}{\leftrightarrow}} \psi_4(p[\alpha \leftarrow p/\alpha\beta]). \tag{8}$$

As $\beta \neq \lambda$, $size(p) \neq size(p[\alpha \leftarrow p/\alpha\beta])$. Note that for any $t \in T_\Delta$, $size(\psi_3(t)) = size(t)$. Hence $\psi_3(p) \neq \psi_3(p[\alpha \leftarrow p/\alpha\beta])$. By Lemma 3.3, $\psi_4(p[\alpha \leftarrow p/\alpha\beta])$'s $\rightarrow_B$-normal form is $\psi_3(p[\alpha \leftarrow p/\alpha\beta])$, and $\psi_4(p)$'s $\rightarrow_B$-normal form is $\psi_3(p)$. Hence

$$(\psi_4(p), \psi_4(p[\alpha \leftarrow p/\alpha\beta])) \notin \overset{*}{\underset{B}{\leftrightarrow}} \cap T_\Sigma \times T_\Sigma.$$

Thus

$$(\psi_4(p), \psi_4(p[\alpha \leftarrow p/\alpha\beta])) \notin \overset{*}{\underset{S}{\leftrightarrow}}$$

implying that

$$(\psi_4(p), \psi_4(p[\alpha \leftarrow \psi_4(p)/\alpha\beta])) \notin \leftrightarrow_R^* \cap \leftrightarrow_S^*,$$

which is a contradiction by (6) and (8). $\square$

The following is a simple consequence of Theorem 4.1.

Theorem 4.2. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. If $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is infinite or $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ is infinite, then there is no gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$.*

5. Case that both tree languages $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ and $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ are finite

In this section we study the case that both $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ and $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ are finite.

Theorem 5.1. *Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. If both $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ and $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ are finite, then we can effectively construct a gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$.*

Proof. We now define tree automaton D over Σ. Let $STATES_D = STATES_A \times STATES_B$. The elements of $STATES_D$ are denoted as $\lfloor a, b \rfloor$. For any $f \in \Sigma_m$, $m \geqslant 0$, $a_1, \ldots, a_m \in STATES_A$, and $b_1, \ldots, b_m \in STATES_B$, we put the rule

$$f(\lfloor a_1, b_1 \rfloor, \ldots, \lfloor a_m, b_m \rfloor) \rightarrow \lfloor a, b \rfloor$$

in D if and only if the rule $f(a_1, \ldots, a_m) \rightarrow a$ is in A and the rule $f(b_1 \ldots, b_m) \rightarrow b$ is in B.

As A and B are deterministic tree automata over Σ, D is deterministic as well.

Lemma 5.2. *For each $p \in T_\Sigma$, $a \in STATES_A$, and $b \in STATES_B$, $p \rightarrow_D^* \lfloor a, b \rfloor$ if and only if $p \rightarrow_A^* a$ and $p \rightarrow_B^* b$.*

Proof. By induction on the height of p. $\square$

Lemma 5.3. (i) *Let* $\lceil L(A,a_1), L(A,a_2), L(B,b)\rceil \in \Gamma^{\leftrightarrow_R^*,\leftrightarrow_S^*}$ *be arbitrary. Then we can effectively construct trees* $tree(a_1,b), tree(a_2,b) \in T_\Sigma$ *such that* $tree(a_1,b) \rightarrow_D^* \lfloor a_1,b_1\rfloor$ *and* $tree(a_2,b) \rightarrow_D^* \lfloor a_2,b_2\rfloor$.

(ii) *Let* $\lceil L(B,b_1), L(B,b_2), L(A,a)\rceil \in \Gamma^{\leftrightarrow_S^*,\leftrightarrow_R^*}$ *be arbitrary. Then we can effectively construct trees* $tree(a,b_1), tree(a,b_2) \in T_\Sigma$ *such that* $tree(a,b_1) \rightarrow_D^* \lfloor a,b_1\rfloor$ *and* $tree(a,b_2) \rightarrow_D^* \lfloor a,b_2\rfloor$.

Proof. Let $\lceil L(A,a_1), L(A,a_2), L(B,b)\rceil \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$ be arbitrary. Then $L(A,a_1)\cap L(B,b) \neq \emptyset$ and $L(A,a_2)\cap L(B,b) \neq \emptyset$. Let $t_1 \in L(A,a_1)\cap L(B,b)$ and $t_2 \in L(A,a_2)\cap L(B,b)$. By Lemma 5.2, $t_1 \rightarrow_D^* \lfloor a_1,b\rfloor$ and $t_2 \rightarrow_D^* \lfloor a_2,b\rfloor$. Hence states $\lfloor a_1,b\rfloor$ and $\lfloor a_2,b\rfloor$ are reachable states of D. By Proposition 2.4, we can effectively construct trees $tree(a_1,b), tree(a_2,b) \in T_\Sigma$ such that $tree(a_1,b) \rightarrow_D^* \lfloor a_1,b\rfloor$ and $tree(a_2,b) \rightarrow_D^* \lfloor a_2,b\rfloor$.

The proof of (ii) is similar to that of (i). □

We now construct gtrs U over Σ in six steps.

Step 1. By Lemma 5.3, for each $\lceil L(A,a_1), L(A,a_2), L(B,b)\rceil \in \Gamma^{\leftrightarrow_R^*,\leftrightarrow_S^*}$, we construct trees $tree(a_1,b), tree(a_2,b) \in T_\Sigma$ such that $tree(a_1,b) \rightarrow_D^* \lfloor a_1,b_1\rfloor$ and $tree(a_2,b) \rightarrow_D^* \lfloor a_2,b_2\rfloor$. By Lemma 5.2, $tree(a_1,b) \in L(A,a_1)\cap L(B,b)$ and $tree(a_2,b) \in L(A,a_2)\cap L(B,b)$.

Step 2. Similarly, for each $\lceil L(B,b_1), L(B,b_2), L(A,a)\rceil \in \Gamma^{\leftrightarrow_S^*,\leftrightarrow_R^*}$, construct trees $tree(a,b_1), tree(a,b_2) \in T_\Sigma$ such that $tree(a,b_1) \rightarrow_D^* \lfloor a,b_1\rfloor$ and $tree(a,b_2) \rightarrow_D^* \lfloor a,b_2\rfloor$. By Lemma 5.2, $tree(a,b_1) \in L(A,a)\cap L(B,b_1)$ and $tree(a,b_2) \in L(A,a)\cap L(B,b_2)$.

Step 3. For each tree $t \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$, we define l_t from t by replacing each leaf $\lceil L(A,a_1), L(A,a_2), L(B,b)\rceil$ by $tree(a_1,b)$. Similarly, for each tree $t \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$, we define r_t from t by replacing each leaf $\lceil L(A,a_1), L(A,a_2), L(B,b)\rceil$ by $tree(a_2,b)$.

Step 4. For each tree $t \in \leftrightarrow_S^* \diamond \leftrightarrow_R^*$, we define l_t from t by replacing each leaf $\lceil L(B,b_1), L(B,b_2), L(A,a)\rceil$ by $tree(a,b_1)$. Similarly, for each tree $t \in \leftrightarrow_S^* \diamond \leftrightarrow_R^*$, we define r_t from t by replacing each leaf $\lceil L(B,b_1), L(B,b_2), L(A,a)\rceil$ by $tree(a,b_2)$.

Step 5. By Proposition 2.17 we effectively construct a reduced gtrs Q over Σ such that $\leftrightarrow_Q^* = \leftrightarrow_D^* \cap T_\Sigma \times T_\Sigma$.

Step 6. Let

$$U = Q \cup \left\{ l_t \rightarrow r_t \mid t \in \underset{R}{\overset{*}{\leftrightarrow}} \diamond \underset{S}{\overset{*}{\leftrightarrow}} \cup \underset{S}{\overset{*}{\leftrightarrow}} \diamond \underset{R}{\overset{*}{\leftrightarrow}} \right\}.$$

We show that

$$\underset{U}{\overset{*}{\leftrightarrow}} = \underset{R}{\overset{*}{\leftrightarrow}} \cap \underset{S}{\overset{*}{\leftrightarrow}}. \tag{9}$$

By the definition of $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ and $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ in Section 2.2 and Conditions (I)–(IV) in Section 2.4, for each tree $t \in \leftrightarrow_R^* \diamond \leftrightarrow_S^* \cup \leftrightarrow_S^* \diamond \leftrightarrow_R^*$, $(l_t, r_t) \in \leftrightarrow_A^* \cap \leftrightarrow_B^* \cap T_\Sigma \times T_\Sigma$. Hence by Conditions (I) and (III) in Section 2.4, $(l_t, r_t) \in \leftrightarrow_R^* \cap \leftrightarrow_S^*$.

Recall that D is a deterministic tree automaton over Σ. By Propositions 2.1 and 2.5, D is a convergent gtrs over the ranked alphabet $\Sigma \cup STATES_D$. Let $(p,q) \in \leftrightarrow_D^* \cap T_\Sigma \times T_\Sigma$.

Then we have $p \rightarrow_D^* u$ and $q \rightarrow_D^* u$ for some $u \in T_{\Sigma \cup STATES_D}$. By Lemma 5.2, $p \rightarrow_A^* u_1$ and $q \rightarrow_A^* u_1$ forsome $u_1 \in T_{\Sigma \cup STATES_A}$, and $p \rightarrow_B^* u_2$ and $q \rightarrow_B^* u_2$ for some $u_2 \in T_{\Sigma \cup STATES_B}$. Hence $\leftrightarrow_D^* \cap T_\Sigma \times T_\Sigma \subseteq \leftrightarrow_A^* \cap \leftrightarrow_B^* \cap T_\Sigma \times T_\Sigma$. Hence $Q \subseteq \leftrightarrow_R^* \cap \leftrightarrow_S^*$. Hence by the definition of U, $U \subseteq \leftrightarrow_R^* \cap \leftrightarrow_S^*$. Thus

$$\overset{*}{\underset{U}{\leftrightarrow}} \subseteq \overset{*}{\underset{R}{\leftrightarrow}} \cap \overset{*}{\underset{S}{\leftrightarrow}}. \tag{10}$$

We show that

$$\overset{*}{\underset{R}{\leftrightarrow}} \cap \overset{*}{\underset{S}{\leftrightarrow}} \subseteq \overset{*}{\underset{U}{\leftrightarrow}}. \tag{11}$$

To this end, we show that

$$\overset{*}{\underset{A}{\leftrightarrow}} \cap \overset{*}{\underset{B}{\leftrightarrow}} \cap T_\Sigma \times T_\Sigma \subseteq \overset{*}{\underset{U}{\leftrightarrow}}. \tag{12}$$

Let $(p,q) \in \leftrightarrow_A^* \cap \leftrightarrow_B^* \cap T_\Sigma \times T_\Sigma$. By Propositions 2.1 and 2.5, A is a convergent gtrs over the ranked alphabet $\Sigma \cup STATES_A$, and B is a convergent gtrs over the ranked alphabet $\Sigma \cup STATES_B$. Hence

$$p = u_A[p^1, \ldots, p^\xi] \overset{*}{\underset{A}{\rightarrow}} u_A[a^{(1)}, \ldots, a^{(\xi)}] \tag{13}$$

and

$$q = u_A[q^1, \ldots, q^\xi] \overset{*}{\underset{A}{\rightarrow}} u_A[a^{(1)}, \ldots, a^{(\xi)}], \tag{14}$$

where $u_A \in \bar{T}_\Sigma(X_\xi)$, $\xi \geqslant 0$, $p^1, \ldots, p^\xi, q^1, \ldots, q^\xi \in T_\Sigma$, $a^{(1)}, \ldots, a^{(\xi)} \in A$, and $u_A[a^{(1)}, \ldots, a^{(\xi)}]$ is the $\rightarrow_A$-normal form of both p and q. Furthermore,

$$p = u_B[r^1, \ldots, r^\omega] \overset{*}{\underset{B}{\rightarrow}} u_B[b^{(1)}, \ldots, b^{(\omega)}] \tag{15}$$

and

$$q = u_B[s^1, \ldots, s^\omega] \overset{*}{\underset{B}{\rightarrow}} u_B[b^{(1)}, \ldots, b^{(\omega)}], \tag{16}$$

where $u_B \in \bar{T}_\Sigma(X_\omega)$, $\omega \geqslant 0$, $r^1, \ldots, r^\omega, s^1, \ldots, s^\omega \in T_\Sigma$, $b^{(1)}, \ldots, b^{(\omega)} \in B$, and $u_B[b^{(1)}, \ldots, b^{(\omega)}]$ is the $\rightarrow_B$-normal form of both p and q. Let $u \in \bar{T}_\Sigma(X_\theta)$, $\theta \geqslant 0$, be such that

- $O(u) = O(u_A) \cap O(u_B)$,
- for each $\alpha \in O(u)$, $lab(u, \alpha) = lab(p, \alpha)$,
- $p = u[p_1, \ldots, p_n, v_1[p_{11}, \ldots, p_{1m_1}], \ldots, v_k[p_{k1}, \ldots, p_{km_k}], v_{k+1}[p_{k+1,1}, \ldots, p_{k+1,m_{k+1}}], \ldots, v_{k+l}[p_{k+l,1}, \ldots, p_{k+l,m_{k+l}}]]$,
- $q = u[q_1, \ldots, q_n, v_1[q_{11}, \ldots, q_{1m_1}], \ldots, v_k[q_{k1}, \ldots, q_{km_k}], v_{k+1}[q_{k+1,1}, \ldots, q_{k+1,m_{k+1}}], \ldots, v_{k+l}[q_{k+l,1}, \ldots, q_{k+l,m_{k+l}}]]$,
- $\theta = n + m_1 + m_2 + \cdots + m_k + m_{k+1} + m_{k+2} + \cdots + m_{k+l}$, $n, m_1, m_2, \ldots, m_k, m_{k+1}, m_{k+2}, \ldots, m_{k+l}, k, l \geqslant 0$,

- $v_i \in \bar{T}_\Sigma(X_{m_i})$ for $1 \leqslant i \leqslant k+l$,
- $p_i \in T_\Sigma$ for $1 \leqslant i \leqslant n$ and $p_{ij} \in T_\Sigma$ for $1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$,
- $q_i \in T_\Sigma$ for $1 \leqslant i \leqslant n$ and $q_{ij} \in T_\Sigma$ for $1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$,
- $u_A = \sigma_A(u[x_{n+k+1} \leftarrow v_{k+1}, \ldots, x_{n+k+l} \leftarrow v_{k+l}])$ for some variable renaming substitution $\sigma_A : X \to X$, and
- $u_B = \sigma_B(u[x_{n+1} \leftarrow v_1, \ldots, x_{n+k} \leftarrow v_k])$ for some variable renaming substitution $\sigma_B : X \to X$.

Then Conditions (B1)–(B9) hold,

(B1) $p_i \to_A^* a_i$ and $p_i \to_B^* b_i$ for some $a_i \in STATES_A$ and $b_i \in STATES_B$ for $1 \leqslant i \leqslant n$,
(B2) $q_i \to_A^* a_i$ and $q_i \to_B^* b_i$ for $1 \leqslant i \leqslant n$,
(B3) $p_{ij} \to_A^* a_{ij}$ and $p_{ij} \to_B^* b_{ij}$ for some $a_{ij} \in STATES_A$ and $b_{ij} \in STATES_B$ for $1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$,
(B4) $q_{ij} \to_A^* a'_{ij}$ and $q_{ij} \to_B^* b_{ij}$ for some $a'_{ij} \in STATES_A$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m_i$,
(B5) $q_{ij} \to_A^* a_{ij}$ and $q_{ij} \to_B^* b'_{ij}$ for some $b'_{ij} \in STATES_B$ for $k+1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$.
(B6) $v_i[a_{i1}, \ldots, a_{im_i}] \to_A^* a^i$ for some $a^i \in STATES_A$ for $1 \leqslant i \leqslant k$,
(B7) $v_i[a'_{i1}, \ldots, a'_{im_i}] \to_A^* a^i$ for some $a^i \in STATES_A$ for $1 \leqslant i \leqslant k$,
(B8) $v_i[b_{i1}, \ldots, b_{im_i}] \to_B^* b^i$ for some $b^i \in STATES_B$ for $k+1 \leqslant i \leqslant k+l$, and
(B9) $v_i[b'_{i1}, \ldots, b'_{im_i}] \to_B^* b^i$ for some $b^i \in STATES_B$ for $k+1 \leqslant i \leqslant k+l$.

Conditions (B1)–(B9) imply (C1) and (C2).

(C1) $p = u[p_1, \ldots, p_n, v_1[p_{11}, \ldots, p_{1m_1}], \ldots, v_k[p_{k1}, \ldots, p_{km_k}],$
$v_{k+1}[p_{k+1,1}, \ldots, p_{k+1,m_{k+1}}], \ldots, v_{k+l}[p_{k+l,1}, \ldots, p_{k+l,m_{k+l}}]] \to_A^*$
$u[a_1, \ldots, a_n, v_1[a_{11}, \ldots, a_{1m_1}], \ldots, v_k[a_{k1}, \ldots, a_{km_k}],$
$v_{k+1}[a_{k+1,1}, \ldots, a_{k+1,m_{k+1}}], \ldots, v_{k+l}[a_{k+l,1}, \ldots, a_{k+l,m_{k+l}}]] \to_A^*$
$u[a_1, \ldots, a_n, a^1, \ldots, a^k, \ldots, v_{k+1}[a_{k+1,1}, \ldots, a_{k+1,m_{k+1}}], \ldots,$
$v_{k+l}[a_{k+l,1}, \ldots, a_{k+l,m_{k+l}}]]$
and
(C2) $q = u[q_1, \ldots, q_n, v_1[q_{11}, \ldots, q_{1m_1}], \ldots, v_k[q_{k1}, \ldots, q_{km_k}],$
$v_{k+1}[q_{k+1,1}, \ldots, q_{k+1,m_{k+1}}], \ldots, v_{k+l}[q_{k+l,1}, \ldots, q_{k+l,m_{k+l}}]] \to_A^*,$
$u[a_1, \ldots, a_n, v_1[a'_{11}, \ldots, a'_{1m_1}], \ldots, v_k[a'_{k1}, \ldots, a'_{km_k}], \ldots,$
$v_{k+1}[a_{k+1,1}, \ldots, a_{k+1,m_{k+1}}], \ldots, v_{k+l}[a_{k+l,1}, \ldots, a_{k+l,m_{k+l}}] \to_A^*$
$u[a_1, \ldots, a_n, a^1, \ldots, a^k, v_{k+1}[a_{k+1,1}, \ldots, a_{k+1,m_{k+1}}], \ldots,$
$v_{k+l}[a_{k+l,1}, \ldots, a_{k+l,m_{k+l}}]].$

Furthermore, (B1)–(B9) imply (C3) and (C4).

(C3) $p = u[p_1, \ldots, p_n, v_1[p_{11}, \ldots, p_{1m_1}], \ldots, v_k[p_{k1}, \ldots, p_{km_k}],$
$v_{k+1}[p_{k+1,1}, \ldots, p_{k+1,m_{k+1}}], \ldots, v_{k+l}[p_{k+l,1}, \ldots, p_{k+l,m_{k+l}}]] \to_B^*$
$u[b_1, \ldots, b_n, v_1[b_{11}, \ldots, b_{1m_1}], \ldots, v_k[b_{k1}, \ldots, b_{km_k}],$
$v_{k+1}[b_{k+1,1}, \ldots, b_{k+1,m_{k+1}}], \ldots, v_{k+l}[b_{k+l,1}, \ldots, b_{k+l,m_{k+l}}]] \to_B^*$
$u[b_1, \ldots, b_n, v_1[b_{11}, \ldots, b_{1m_1}], \ldots, v_k[b_{k1}, \ldots, b_{km_k}], b^{k+1}, \ldots, b^{k+l}]$
and
(C4) $q = u[q_1, \ldots, q_n, v_1[q_{11}, \ldots, q_{1m_1}], \ldots, v_k[q_{k1}, \ldots, q_{km_k}],$
$v_{k+1}[q_{k+1,1}, \ldots, q_{k+1,m_{k+1}}], \ldots, v_{k+l}[q_{k+l,1}, \ldots, q_{k+l,m_{k+l}}]] \to_B^*,$
$u[b_1, \ldots, b_n, v_1[b_{11}, \ldots, b_{1m_1}], \ldots, v_k[b_{k1}, \ldots, b_{km_k}],$
$v_{k+1}[b'_{k+1,1}, \ldots, b'_{k+1,m_{k+1}}], \ldots, v_{k+l}[b'_{k+l,1}, \ldots, b'_{k+l,m_{k+l}}]] \to_B^*$
$u[b_1, \ldots, b_n, v_1[b_{11}, \ldots, b_{1m_1}], \ldots, v_k[b_{k1}, \ldots, b_{km_k}], b^{k+1}, \ldots, b^{k+l}].$

Hence Conditions (D1)–(D11) hold as well.

(D1) $p_i \to_D^* \lfloor a_i, b_i \rfloor$ for some $a_i \in STATES_A$ and $b_i \in STATES_B$ for $1 \leqslant i \leqslant n$.

(D2) $q_i \to_D^* \lfloor a_i, b_i \rfloor$ for $1 \leqslant i \leqslant n$.

(D3) $p_{ij} \to_D^* \lfloor a_{ij}, b_{ij} \rfloor$ for some $a_{ij} \in STATES_A$ and $b_{ij} \in STATES_B$ for $1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$.

(D4) $q_{ij} \to_D^* \lfloor a'_{ij}, b_{ij} \rfloor$ for some $a'_{ij} \in STATES_A$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m_i$.

(D5) $q_{ij} \to_D^* \lfloor a_{ij}, b'_{ij} \rfloor$ for some $b'_{ij} \in STATES_B$ for $k+1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$.

(D6) $v_i[\lceil L(A, a_{i1}), L(A, a'_{i1}), L(B, b_{i1}) \rceil, \ldots, \lceil L(A, a_{im_i}), L(A, a'_{im_i}), L(B, b_{im_i}) \rceil] \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$ for $1 \leqslant i \leqslant k$.

(D7) $v_i[\lceil L(B, b_{i1}), L(B, b'_{i1}), L(A, a_{i1}) \rceil, \ldots, \lceil L(B, b_{im_i}), L(B, b'_{im_i}), L(A, a_{im_i}) \rceil] \in \leftrightarrow_S^* \diamond \leftrightarrow_R^*$ for $k+1 \leqslant i \leqslant k+l$.

Hence Conditions (E1)–(E6) hold as well.

(E1) $p_i \to_U^* tree(a_i, b_i)$ for $1 \leqslant i \leqslant n$.

(E2) $q_i \to_U^* tree(a_i, b_i)$ for $1 \leqslant i \leqslant n$.

(E3) $p_{ij} \to_U^* tree(a_{ij}, b_{ij})$ for $1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$.

(E4) $q_{ij} \to_U^* tree(a'_{ij}, b_{ij})$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m_i$.

(E5) $q_{ij} \to_U^* tree(a_{ij}, b'_{ij})$ for $k+1 \leqslant i \leqslant k+l$, $1 \leqslant j \leqslant m_i$.

(E6) $v_i[\lceil L(A, a_{i1}), L(A, a'_{i1}), L(B, b_{i1}) \rceil, \ldots, \lceil L(A, a_{im_i}), L(A, a'_{im_i}), L(B, b_{im_i}) \rceil] \in \leftrightarrow_R^* \diamond \leftrightarrow_S^*$ for $1 \leqslant i \leqslant k$.

(E7) $v_i[\lceil L(B, b_{i1}), L(B, b'_{i1}), L(A, a_{i1}) \rceil, \ldots, \lceil L(B, b_{im_i}), L(B, b'_{im_i}), L(A, a_{im_i}) \rceil] \in \leftrightarrow_S^* \diamond \leftrightarrow_R^*$ for $k+1 \leqslant i \leqslant k+l$.

Note that (E6), (E7) are symmetrical. By (E6) we have (E8).

(E8) the rule $v_i[tree(a_{i1}, b_{i1}), \ldots, tree(a_{im_i}, b_{im_i})] \to v_i[tree(a'_{i1}, b_{i1}), \ldots, tree(a'_{im_i}, b_{im_i})]$ is in U for $1 \leqslant i \leqslant k$.

By (E7) we have (E9).

(E9) the rule $v_i[tree(a_{i1}, b_{i1}), \ldots, tree(a_{im_i}, b_{im_i})] \to v_i[tree(a_{i1}, b'_{i1}), \ldots, tree(a_{im_i}, b'_{im_i})]$ is in U for $k+1 \leqslant i \leqslant k+l$.

Conditions (E8) and (E9) are symmetrical as well. Conditions (E10) and (E11) are the summary of (E1)–(E9).

(E10)
$$\begin{aligned}
p \to_U^* {} & u[tree(a_1, b_1), \ldots, tree(a_n, b_n), \\
& v_1[tree(a_{11}, b_{11}), \ldots, tree(a_{1m_1}, b_{1m_1})], \ldots, \\
& v_k[tree(a_{k1}, b_{k1}), \ldots, tree(a_{km_k}, b_{km_k})], \\
& v_{k+1}[tree(a_{k+1,1}, b_{k+1,1}), \ldots, tree(a_{k+1m_{k+1}}, b_{k+1m_{k+1}})], \ldots, \\
& v_{k+l}[tree(a_{k+l,1}, b_{k+l,1}), \ldots, tree(a_{k+lm_{k+l}}, b_{k+lm_{k+l}})]] \to_U^* \\
& u[tree(a_1, b_1), \ldots, tree(a_n, b_n), \\
& v_1[tree(a'_{11}, b_{11}), \ldots, tree(a'_{1m_1}, b_{1m_1})], \ldots, \\
& v_k[tree(a'_{k1}, b_{k1}), \ldots, tree(a'_{km_k}, b_{km_k})], \\
& v_{k+1}[tree(a_{k+1,1}, b'_{k+1,1}), \ldots, tree(a_{k+1m_{k+1}}, b'_{k+1m_{k+1}})], \ldots, \\
& v_{k+l}[tree(a_{k+l,1}, b'_{k+l,1}), \ldots, tree(a_{k+lm_{k+l}}, b'_{k+lm_{k+l}})]]
\end{aligned}$$

(E11)
$$\begin{aligned}
q \to_U^* {} & u[tree(a_1, b_1), \ldots, tree(a_n, b_n), \\
& v_1[tree(a'_{11}, b_{11}), \ldots, tree(a'_{1m_1}, b_{1m_1})], \ldots, \\
& v_k[tree(a'_{k1}, b_{k1}), \ldots, tree(a'_{km_k}, b_{km_k})], \\
& v_{k+1}[tree(a_{k+1,1}, b'_{k+1,1}), \ldots, tree(a_{k+1m_{k+1}}, b'_{k+1m_{k+1}})], \ldots, \\
& v_{k+l}[tree(a_{k+l,1}, b'_{k+l,1})\ tree(a_{k+lm_{k+l}}, b'_{k+lm_{k+l}})]].
\end{aligned}$$

Conditions (E10) and (E11) imply that $p \leftrightarrow_U^* q$. Thus Condition (12) holds. Hence by Conditions (I) and (III) in Section 2.4, Condition (11) holds. By Condition (10), Condition (9) holds as well. $\square$

6. Decidability result

Let R and S be arbitrary ground term rewrite systems over a ranked alphabet Σ. By Theorem 3.8, it is decidable if both $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ and $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ are finite. If $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ is infinite or $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ is infinite, then by Theorem 4.1 there is no gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$. If both $\leftrightarrow_R^* \diamond \leftrightarrow_S^*$ and $\leftrightarrow_S^* \diamond \leftrightarrow_R^*$ are finite, then by Theorem 5.1 we can effectively construct a gtrs U such that $\leftrightarrow_R^* \cap \leftrightarrow_S^* = \leftrightarrow_U^*$. Hence we have shown our main decidability result.

Theorem 6.1. *It is decidable for any given ground term rewrite systems R and S if there is a ground term rewrite system U such that $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^*$. If the answer is yes, then we can effectively construct such a ground term rewrite system U.*

It is still open how we can generalize our results for three ground term rewrite systems. Let R, S, and Q be arbitrary ground term rewrite systems over Σ. When constructing the generalization $\Gamma^{\leftrightarrow_R^*, \leftrightarrow_S^*, \leftrightarrow_Q^*}$ of the alphabet $\Gamma^{\leftrightarrow_R^*, \leftrightarrow_S^*}$, we face the problem that $trunk(\leftrightarrow_R^*)$, $trunk(\leftrightarrow_S^*)$, and $trunk(\leftrightarrow_Q^*)$ are pairwise different. Hence it is open how we can introduce tree language $\leftrightarrow_R^* \diamond \leftrightarrow_S^* \diamond \leftrightarrow_Q^*$. We state this question as a conjecture.

Conjecture 6.2. *It is decidable for any given ground term rewrite systems R, S, and Q if there is a ground term rewrite system U such that $\leftrightarrow_U^* = \leftrightarrow_R^* \cap \leftrightarrow_S^* \cap \leftrightarrow_Q^*$. If the answer is yes, then we can effectively construct such a ground term rewrite system U.*

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